A LOW RANK PROPERTY AND NONEXISTENCE OF HIGHER DIMENSIONAL HORIZONTAL SOBOLEV SETS

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Abstract. We establish a "low rank property" for Sobolev mappings that pointwise solve a first order nonlinear system of PDEs, whose smooth solutions have the so-called "contact property". As a consequence, Sobolev mappings from an open set of the plane, taking values in the first Heisenberg group $\mathbb{H}^1$ and that have almost everywhere maximal rank must have images with positive 3-dimensional Hausdorff measure with respect to the sub-Riemannian distance of $\mathbb{H}^1$. This provides a complete solution to a question raised in a paper by Z. M. Balogh, R. Hoefer-Isenegger and J. T. Tyson. Our approach differs from the previous ones. Its technical aspect consists in performing an "exterior differentiation by blow-up", where the standard distributional exterior differentiation is not possible. This method extends to higher dimensional Sobolev mappings, taking values in higher dimensional Heisenberg groups.

Contents

2010 *Mathematics Subject Classification.* Primary: 46E35, 53C17. Secondary: 26B20, 22E25.
Key words and phrases. Heisenberg group, nonisotropic submanifolds, Hausdorff measure.
Both the first and third author acknowledge the European Project ERC AdG *GeMeThNES*.
The final publication is available at Springer via DOI 10.1007/s12220-014-9478-1.

1. Introduction

A k-dimensional *tangent distribution* in $\mathbb{R}^n$ can be seen as a family of k-dimensional planes that are locally spanned by k linearly independent smooth vector fields. When all the tangent spaces of a k-dimensional submanifold Σ coincide with these k-dimensional planes, we say that Σ is an *integral submanifold* of the distribution.

The classical Frobenius theorem characterizes those tangent distributions that give a local foliation of the space into families of integral submanifolds. These special tangent distributions are called *involutive*, namely, the Lie brackets of those vector fields that linearly span the tangent distribution still belong to the same distribution.

An important example of tangent distribution in $\mathbb{R}^3$ is given by the vector fields

$$X_1 = \partial_{x_1} - x_2\partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} \,. \tag{1.1}$$

The vector fields X_1, X_2 and X_3 generate the Lie algebra of the 3-dimensional Heisenberg group $\mathbb{H}^1$, where $X_3 = \frac{1}{2}[X_1, X_2] = \partial_{x_3}$. For convenience, we mention the group operation of $\mathbb{H}^1$: it is given on $\mathbb{R}^3$ by

$$x \cdot y = x + y + (0, 0, x_1y_2 - x_2y_1), \tag{1.2}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. Precisely, the vector fields X_1, X_2 and X_3 are left invariant with respect to the group operation (1.2). The tangent distribution defined by (1.1) is the so-called *horizontal distribution* and it can be identified with the *horizontal subbundle* $H\mathbb{H}^1$. Integral submanifolds of the horizontal distribution can be equivalently considered to be "tangent" to $H\mathbb{H}^1$. The tangent distribution (1.1) also determines an intrinsic length distance called the *sub-Riemannian distance*, or, in short, the *SR-distance*. We refer to [18] for more information on this notion.

Since the horizontal distribution of (1.1) is nowhere involutive, the Frobenius theorem gives the nonexistence of smooth surfaces in $\mathbb{R}^3$ that are tangent to $H\mathbb{H}^1$. However, one may still wonder whether there exist more general "2-dimensional sets" that can be still considered "tangent" to this distribution in a broad sense. This problem is amazingly related to the study of the Hausdorff dimension of sets with respect to the sub-Riemannian distance. In this connection, Z. M. Balogh and J. T. Tyson have constructed an interesting example of "horizontal fractal", called the *Heisenberg square* Q_H, [3]. The 2-dimensional Hausdorff measure of Q_H with respect to both the SR-distance and the Euclidean distance is finite and positive, see [3, Theorem 1.10]. As proved in [4], it is possible to find a BV function $g : (0,1)^2 \to \mathbb{R}$, whose graph G is contained in Q_H and satisfies

$$0 < \mathcal{H}^2_d(G) < +\infty. \tag{1.3}$$

The symbol $\mathcal{H}^2_d$ denotes the Hausdorff measure with respect to the SR-distance d of $\mathbb{H}^1$. Condition (1.3) never holds for graphs of smooth functions. It can be interpreted as a "metric definition" of horizontality for lower regular sets. In fact, in the general Heisenberg group $\mathbb{H}^n$, represented by $\mathbb{R}^{2n+1}$ equipped with the left invariant vector fields

$$X_i = \partial_{x_i} - x_{i+n}\partial_{x_{2n+1}}, \quad X_{n+i} = \partial_{x_{n+i}} + x_i\partial_{x_{2n+1}} \quad \text{and} \quad i = 1, \dots, n, \tag{1.4}$$

spanning $H\mathbb{H}^n$, every C^1 smooth m-dimensional submanifold $\Sigma \subset \mathbb{H}^n$ that is everywhere tangent to $H\mathbb{H}^n$ must have the measure $\mathcal{H}^m_d \llcorner \Sigma$ locally finite. On the other hand, from

Contact Topology, it is well known that not only hypersurfaces but rather all sufficiently smooth submanifolds $\Sigma \subset \mathbb{H}^n$ of dimension m, with $n < m \le 2n$, cannot be everywhere tangent to $H\mathbb{H}^n$, in short $T\Sigma \not\subseteq H\mathbb{H}^n$, see for instance [9, Proposition 1.5.12]. Thus, when $m > n$ there must exist at least one point $x \in \Sigma$ such that $T_x\Sigma \not\subseteq H_x\mathbb{H}^n$.

This fact has an important consequence on the Hausdorff dimension of Σ with respect to the SR-distance d. In fact, when Σ is C^1 smooth, in view of a general negligibility result, [13], joined with area-type formulae, [8, 14, 15], the measure $\mathcal{H}^{m+1}_d \llcorner \Sigma$ has an integral representation with respect to $\mathcal{H}^m_{|\cdot|} \llcorner \Sigma$ and the integrand is proportional to the length of a suitable m-vector. This m-vector is the "vertical tangent m-vector", denoted by $\tau_{\Sigma,\mathcal{V}}$, and it defined as the projection of the unit tangent m-vector of Σ onto the orthogonal subspace to the linear space $\Lambda_m(H\mathbb{H}^n)$ of horizontal m-vectors, see [14] for more details.

The point is that the vertical tangent m-vector does not vanish precisely at all points $x \in \Sigma$ such that $T_x\Sigma \not\subseteq H_x\mathbb{H}^n$. As a consequence, for each smooth m-dimensional submanifold $\Sigma \subset \mathbb{H}^n$ with $m > n$, there holds

$$\mathcal{H}^{m+1}_d(\Sigma) > 0 \,. \tag{1.5}$$

In the case $n = 1$ and $m = 2$, the non-horizontality condition (1.5) for nonsmooth sets has been shown in [4], where Σ is a 2-dimensional Lipschitz graph of $\mathbb{H}^1$. Here the authors raise the interesting question on the existence of horizontal sets in the sense of (1.3) having regularity between Lipschitz and BV. A first answer to this question is given in [16], where it is shown that if Σ is either a 2-dimensional $W^{1,1}_{\mathrm{loc}}$-Sobolev graph or the image of a mapping in $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$ with $p \ge 4/3$ and whose weak differential has almost everywhere maximal rank, then (1.5) must hold with $m = 2$.

The starting point is that a smooth parametrization $f : \Omega \to \Sigma$ satisfies the equation

$$df^3 = f^1 df^2 - f^2 df^1 \tag{1.6}$$

at those points $y \in \Omega$ such that $T_{f(y)}\Sigma \subset H\mathbb{H}^1$, where $\Omega \subset \mathbb{R}^2$ is an open set. In other words, precisely at these points f has the so-called *contact property*, namely, its differential preserves the "horizontal directions". In fact, in the source space all directions are horizontal and in the target these directions are spanned by the vector fields (1.1). The differential obstruction in the everywhere validity of equation (1.6) is easily seen by performing its exterior differentiation, since this implies that the rank of the Jacobian matrix of f is everywhere less than two and this conflicts with the starting assumptions.

Once it is proved that (1.6) cannot hold a.e., then the Whitney extension theorem yields a C^1 smooth submanifold $\tilde{\Sigma}$ that coincides with the original Σ on some measurable subset $A \subset \tilde{\Sigma} \cap \Sigma$ of positive Euclidean surface measure, where in addition $TA \not\subseteq H\mathbb{H}^1$. As a consequence, in view of the previous comments on the density of $\mathcal{H}^3_d \llcorner \tilde{\Sigma}$, we achieve

$$\mathcal{H}^3_d(\Sigma) \ge \mathcal{H}^3_d(A) > 0 \,.$$

In sum, the main technical point consists in being able to accomplish a suitable exterior differentiation of (1.6) under the condition that the regularity of f is as weak as possible. When $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{H}^1)$ an exterior distributional differentiation of (1.6) is possible under the regularity condition $p \ge 4/3$, but this approach does not work for $1 \le p < 4/3$, where the question was left open. The following theorem answers this question.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be open, let $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^3)$ be such that the Jacobian matrix of f has almost everywhere maximal rank and define $\Sigma = f(\Omega)$. It follows that $\mathcal{H}^3_d(\Sigma) > 0$.*

This provides the full answer to the question raised in [4]. Our approach to establish Theorem 1.1 differs from the previous ones and it also works for any higher dimensional Heisenberg group $\mathbb{H}^n$, which we identify with the linear space $\mathbb{R}^{2n+1}$.

Let us summarize the main steps. We consider a mapping $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^{2n+1})$, where Ω is an open set of $\mathbb{R}^m$, then the contact property of equation (1.6) is replaced by

$$df^{2n+1} = \sum_{j=1}^{n} \left(f^j df^{j+n} - f^{j+n} df^j \right) . \tag{1.7}$$

Next, we restrict ourselves to the special case $m = 2$, since it suffices to describe the core of our method. We denote by Df the Jacobian matrix of f, whose columns correspond to the weak derivatives of f. Then we rescale f at Lebesgue points $z \in \Omega$ of both f and Df. Then the rescaled functions $f_{z,\rho}$, introduced in Definition 4.1, are defined on the unit ball $\mathbb{B}$ of $\mathbb{R}^2$ for all $\rho > 0$ sufficiently small and converge to the linear mapping $u : y \mapsto Df(z) \cdot y$ in $W^{1,1}(\mathbb{B})$ as $\rho \to 0^+$. If by contradiction we assume that (1.7) holds a.e., then the one-form

$$\sum_{j=1}^{n} \left(f^j_{z,\rho}\, df^{j+n}_{z,\rho} - f^{j+n}_{z,\rho}\, df^j_{z,\rho} \right) \tag{1.8}$$

is "weakly exact" in the sense that it is a.e. equal to dw_ρ for some $w_\rho \in W^{1,1}(\mathbb{B})$, see Lemma 4.1. We exploit this fact by integrating (1.8) on the Euclidean sphere $\partial B(0, r)$ for almost every $r \in (0, 1)$ and pass to the limit with respect to ρ as it goes to zero by a suitable positive infinitesimal sequence (ρ_k). Since the limit has the form

$$\sum_{j=1}^{n} \left(u^j\, du^{j+n} - u^{j+n}\, du^j \right)$$

with $u(y) = Df(z) \cdot y$ and its oriented integral vanishes on almost every sphere of radius $r \in (0, \rho)$, it follows that

$$\sum_{j=1}^{n} df^j(z) \wedge df^{j+n}(z) = 0\,, \tag{1.9}$$

by the classical Stokes theorem. If $m > 2$, then we obtain (1.9) by a slicing argument, so that the whole range $m \geq 2$ is provided. Joining Lemma 6.1 with (1.9), we deduce that the rank of $Df(z)$ is less than $n + 1$. According to Theorem 6.1, this shows that Sobolev mappings that a.e. satisfy the horizontality condition (1.7) must satisfy a.e. a "low rank property", namely the rank of Df is a.e. less than $n + 1$.

In this connection, we wish to mention that in the first preprint of our paper, [17], we have obtained this low rank result for $W^{1,p(n)}$ functions in Heisenberg groups $\mathbb{H}^n$, where $p(n) = 1$ only for $n = 1$. Then we have been informed that Z. M. Balogh, P. Hajłasz and K. Wildrick have independently obtained this result, with a different approach, [5]. They first observed that the best possible Sobolev class $W^{1,1}$ is available, regardless of the dimension of $\mathbb{H}^n$. Thanks to their announcement, we immediately realized that a slight modification of our method also gives $p(n) = 1$ for all positive $n \in \mathbb{N}$, as in the current version of Theorem 6.1.

Let us point out that in the case $m \le n$, the mapping f can be a smooth embedding that is also a contact mapping, namely, $df(y)(\mathbb{R}^m) \subset H_{f(y)}\mathbb{H}^n$ for every $y \in \Omega$. To see this, it suffices to take local parametrizations of isotropic submanifolds when $\mathbb{H}^n$ is regarded as a contact manifold, [9]. In higher dimensions, the application of Theorem 6.1 appears in the case $n+1 \le m \le 2n$, where it should be seen somehow as a "differential obstruction". It is worth to compare this obstruction with the "Lipschitz obstructions" appearing in the study of Lipschitz homotopy groups of the Heisenberg group, [6]. Our main application of Theorem 6.1 is the following result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set, let $n < m \le 2n$ and let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$. Suppose that the Jacobian matrix Df has rank equal to m almost everywhere and set $\Sigma = f(\Omega)$. Then $\mathcal{H}^{m+1}_d(\Sigma) > 0$.*

This theorem also includes Theorem 1.1. In ending, we wish to point out a curious observation on the graph G of the BV function g mentioned above. We can translate the metric horizontality of (1.3) into a "tangential condition". In fact, one can easily observe that the approximate differential of the graph mapping $f = (x_1, x_2, g)$ must satisfy (1.6) almost everywhere, hence $\operatorname{ap}\nabla g = (-x_2, x_1)$ almost everywhere, see Theorem 6.2. This can be seen as a tangential condition in the sense of Geometric Measure Theory.

Finally, we introduce some notation that will be used throughout the paper. The Euclidean open ball in $\mathbb{R}^m$ with center at x and radius r is denoted by $B(x,r)$. The unit ball $B(0,1)$ will be simply denoted by $\mathbb{B}$. If u is a Sobolev function on an open set of $\mathbb{R}^m$, then du denotes the measurable one form $\sum_{j=1}^k \partial_{x_j} u\, dx_j$, where every coefficient $\partial_{x_j} u$ is the j-th distributional derivative of u. The symbol ∇u denotes the vector of weak derivatives $(\partial_{x_1} u, \dots, \partial_{x_k} u)$, that a.e. coincides with the vector of approximate partial derivatives. The Jacobi matrix of weak derivatives of a vector-valued function f is denoted by Df.

2. Slicing

For the reader's convenience, in this section we recall some well known facts about Sobolev sections, that will be used in the subsequent part of the paper. Let m be a positive integer and denote by $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ the canonical basis of $\mathbb{R}^m$. If $\Gamma \subset \{1, \dots, m\}$ is a set of indices, then V_Γ is the linear span of $\{\boldsymbol{e}_j : j \in \Gamma\}$ and $V_\Gamma^\perp$ is the linear span of $\{\boldsymbol{e}_j : j \in \{1, \dots, m\} \setminus \Gamma\}$. We introduce the orthogonal projections

$$\pi_\Gamma(x) = \sum_{j \in \Gamma} x_j \boldsymbol{e}_j \quad \text{and} \quad \hat{\pi}_\Gamma(x) = x - \pi_\Gamma(x)$$

where $x \in \mathbb{R}^m$, $\pi_\Gamma : \mathbb{R}^m \to V_\Gamma$ and $\hat{\pi}_\Gamma : \mathbb{R}^m \to V_\Gamma^\perp$. Let Q be an open m-dimensional interval in $\mathbb{R}^m$, namely the product of m open intervals, and fix a nonempty subset $\Gamma \subsetneq \{1, \dots, m\}$. We define the projected intervals

$$Q_\Gamma = \pi_\Gamma(Q) \quad \text{and} \quad \hat{Q}_\Gamma = \hat{\pi}_\Gamma(Q).$$

If $u\colon Q \to \mathbb{R}$ is a function and $z \in \hat{Q}_\Gamma$, we define the *section* $u^z : Q_\Gamma \to \mathbb{R}$ as

$$u^z(y) = u(z+y), \qquad y \in Q_\Gamma.$$

Definition 2.1. We say that a sequence $\{u_h\}$ in a Banach space $(X, \|\cdot\|)$ converges *fast* to $u \in X$, or that it is *fast convergent*, if $\sum_{h=1}^\infty \|u_h - u\| < \infty$.

We wish to point out that the fast convergence in $W^{1,1}$ is just the joint fast convergence in L^1 of functions and their gradients. As a consequence of both Fubini's theorem and Beppo Levi's convergence theorem for series, we get the next proposition.

Proposition 2.1. *Let $\{u_h\} \subset W^{1,1}(Q)$ be a sequence which converges fast to $u \in W^{1,1}(Q)$. Then for each $k = 1, \dots, m$ and for almost every $z \in \hat{Q}_\Gamma$ we have $u^z, (\partial_{y_k} u)^z, u_h^z, (\partial_{y_k} u_h)^z \in L^1(Q_\Gamma)$, $h \in \mathbb{N}$, further, $\{u_h^z\}$ converges fast to u^z in $L^1(Q_\Gamma)$ and $(\partial_{y_k} u_h)^z$ converges fast to $(\partial_{y_k} u)^z$ in $L^1(Q_\Gamma)$.*

Each $u \in W^{1,1}(Q)$ is a limit of a fast convergent sequence of smooth functions. Applying Proposition 2.1 we obtain the following consequence.

Proposition 2.2. *Let $u \in W^{1,1}(Q)$. Then for almost every $z \in \hat{Q}_\Gamma$ we have $u^z \in W^{1,1}(Q_\Gamma)$ and*

$$\partial_{y_k} u^z = (\partial_{y_k} u)^z \quad \text{a.e. in } Q_\Gamma, \qquad k = 1, \dots, m. \tag{2.1}$$

Summarizing Propositions 2.1 and 2.2 we obtain the following.

Proposition 2.3. *Let $\{u_h\} \subset W^{1,1}(Q)$ be a sequence which converges fast to $u \in W^{1,1}(Q)$. Then for almost every $z \in \hat{Q}_\Gamma$ we have $u^z, u_h^z \in W^{1,1}(Q_\Gamma)$, $h \in \mathbb{N}$, and $\{u_h^z\}$ converges fast to u^z in $W^{1,1}(Q_\Gamma)$.*

3. Oriented integration on the circle

The idea of slicing can be also applied to study the behavior of Sobolev functions on a.e. sphere. However, for our purposes it is enough to perform this analysis in $\mathbb{R}^2$ only, so that we will study Sobolev spaces on circles.

Definition 3.1 (Function spaces on the circle)**.** Consider the circle $\partial B(x,r)$ and its parametrization

$$\psi(t) = (x_1 + r\cos t,\, x_2 + r \sin t), \qquad t \in \mathbb{R}. \tag{3.1}$$

We define $\psi_- = \psi\lfloor_{(-\pi,\pi)}$ and $\psi_+ = \psi\lfloor_{(0,2\pi)}$, hence (ψ_+, ψ_-) is an oriented atlas of $\partial B(x,r)$. This atlas automatically defines function spaces on $\partial B(x,r)$. Let X be a generic function space symbol which may refer e.g. to $W^{1,p}$, L^p or C. We say that $u : \partial B(x,r) \to \mathbb{R}$ belongs to $X(\partial B(x,r))$ if $u \circ \psi_-$ belongs to $X\big((-\pi,\pi)\big)$ and $u \circ \psi_+$ belongs to $X\big((0,2\pi)\big)$.

Definition 3.2 (Integrable forms on the circle)**.** Let us consider $u, v : \partial B(x,r) \to \mathbb{R}$. Then the *oriented integral* of the differential form $u\,dv$ is defined as follows

$$\int_{\partial B(x,r)} u\,dv = \int_{-\pi}^{\pi} (u\circ\psi)(t)\,(v\circ\psi)'(t)\,dt,$$

whenever this expression makes sense, if e.g. $u \in L^\infty(\partial B(x,r))$, $v \in W^{1,1}(\partial B(x,r))$ and $(v\circ\psi)'$ is the distributional derivative of $v \circ \psi$.

The following lemma relates the fast convergence with the convergence of oriented integrals on spherical sections.

Lemma 3.1. *Let $u, u_h, v, v_h \in W^{1,1}(B(x,\rho))$, $h \in \mathbb{N}$, and suppose that both $u_h \to u$ and $v_h \to v$ fast in $W^{1,1}(B(x,\rho))$. Then for almost every $0 < r < \rho$ the restrictions of u, u_h, v, v_h to $\partial B(x,r)$ belong to $W^{1,1}(\partial B(x,r))$ and*

$$\int\limits_{\partial B(x,r)} u_h \, dv_h \to \int\limits_{\partial B(x,r)} u \, dv \,. \tag{3.2}$$

Proof. We use the polar coordinates associated to the mapping

$$\Psi(r,t) = (x_1 + r\cos t,\ x_2 + r\sin t).$$

The mapping $\Psi^r = \Psi(r,\cdot)$ will be defined on $(-2\pi, 2\pi)$ for each $r \in (0,\rho)$. First, we observe that given $w \in W^{1,1}(B(x,\rho))$, then $w \circ \Psi$ belongs to $W^{1,1}((\delta,\rho) \times (-2\pi,2\pi))$ for each $\delta \in (0,\rho)$. The fast convergence of both $\{u_h\}$ and $\{v_h\}$ in $W^{1,1}(B(x,r))$ implies that $u_h \circ \Psi$ and $v_h \circ \Psi$ are fast convergent in $W^{1,1}\big((\delta,\rho) \times (-2\pi,2\pi)\big)$ with limits equal to $u \circ \Psi$ and $v \circ \Psi$, respectively. By Proposition 2.3, for a.e. $r \in (\delta,\rho)$ we have that $u_h \circ \Psi^r, v_h \circ \Psi^r, u \circ \Psi^r, v \circ \Psi^r \in W^{1,1}\big((-2\pi,2\pi)\big)$ and both $u_h \circ \Psi^r$ and $v_h \circ \Psi^r$ converge fast in $W^{1,1}\big((-2\pi,2\pi)\big)$ to $u \circ \Psi^r$ and $v \circ \Psi^r$, respectively.

Fix such a good radius r. Then $u \circ \Psi^r$, $u_h \circ \Psi^r$ are absolutely continuous up to a modification on a null set. Using the one-dimensional Sobolev embedding and passing to absolutely continuous representatives, we obtain a uniform convergence $u_h \circ \Psi^r \to u \circ \Psi^r$. Joining with the L^1-convergence $(v_h \circ \Psi^r)' \to (v \circ \Psi^r)'$ we conclude that

$$\begin{aligned}\int_{\partial B(x,r)} u_h \, dv_h &= \int_{-\pi}^{\pi} (u_h \circ \Psi^r)(t)(v_h \circ \Psi^r)'(t)\, dt \to \int_{-\pi}^{\pi} (u \circ \Psi^r)(t)(v \circ \Psi^r)'(t)\, dt \\ &= \int_{\partial B(x,r)} u \, dv\end{aligned}$$

as required. By the arbitrary choice of $\delta > 0$, we have proved that (3.2) holds for a.e. $r \in (0,\rho)$. □

Lemma 3.2. *Let $v \in W^{1,1}(B(x,\rho))$. For almost every $r \in (0,\rho)$, the oriented integral $\int\limits_{\partial B(x,r)} dv$ is well defined and equal to zero.*

Proof. Again, we use the polar coordinates as in the preceding proof. By Proposition 2.2, for a.e. $r \in (0,\rho)$, the section $v \circ \Psi^r$ belongs to $W^{1,1}(-2\pi,2\pi)$. If $\bar{v} \circ \Psi^r$ is the absolutely continuous representative of $v \circ \Psi^r$, we have

$$\int_{\partial B(x,r)} dv = \int_{-\pi}^{\pi} (v \circ \Psi^r)'(t)\, dt = \bar{v} \circ \Psi^r(\pi) - \bar{v} \circ \Psi^r(-\pi) = 0,$$

as $\bar{v} \circ \Psi^r$ is obviously 2π-periodic. □

4. An exterior differentiation by blow up

Throughout this section, we fix an open set $\Omega \subset \mathbb{R}^2$, a mapping $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^{2n+1})$ and a point $z \in \Omega$ that is a Lebesgue point of both f and Df. Recall that z is a Lebesgue point for a measurable function $u \in L^1_{\rm loc}(\Omega)$ if

$$\lim_{r\to 0^+} r^{-2} \int_{B(z,r)} |u(y) - u(z)| \, dy = 0$$

and that almost every point of Ω is a Lebesgue point of u. As already pointed out in the introduction, $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ equipped with the vector fields of (1.4). We fix $\rho > 0$ such that $\overline{B(z,\rho)} \subset \Omega$. $\mathbb{B}$ denotes the open unit ball of $\mathbb{R}^2$ centered at the origin.

Definition 4.1. Let $0 < r \le \rho$ and define the *rescaled function* $f_{z,r} : \mathbb{B} \to \mathbb{R}^{2n+1}$ as

$$f_{z,r}(y) := \frac{f(z+ry) - f(z)}{r} \; .$$

Obviously, $f_{z,r} \in W^{1,1}(\mathbb{B}, \mathbb{R}^{2n+1})$ is well defined whenever $0 < r \le \rho$. We use the assumption that z is a Lebesgue point of both f and Df to conclude that

$$\lim_{r\to 0^+} \int_{\mathbb{B}} |f_{z,r}(y) - Df(z)\cdot y| \, dy = 0, \tag{4.1}$$

cf. e.g. [19, Theorem 3.4.2]. The next lemma provides us with important information on the rescaled function $f_{z,\rho}$.

Lemma 4.1. *If* (1.7) *holds almost everywhere, then whenever* $0 < r < \rho$ *there exists* $w \in W^{1,1}(\mathbb{B})$ *such that*

$$dw(y) = \sum_{j=1}^{n} f^j_{z,r}(y) df^{j+n}_{z,r}(y) - f^{j+n}_{z,r}(y)\, df^j_{z,r}(y) \quad \textit{for a.e.} \quad y \in \mathbb{B}\,,$$

where $w = r^{-1}\Big(f^{2n+1}_{z,r} - \sum_{j=1}^n f^j(z)\, f^{j+n}_{z,r} - f^{j+n}(z)\, f^j_{z,r}\Big)$.

Proof. From the a.e. validity of (1.7) joined with basic properties of weak derivatives of Sobolev functions, we get

$$\nabla f^{2n+1}_{z,r}(y) = \nabla f^{2n+1}(z+ry) = \sum_{j=1}^n f^j(z+ry)\nabla f^{j+n}(z+ry) - f^{j+n}(z+ry)\nabla f^j(z+ry)$$

for a.e. $y \in \mathbb{B}$. We add and subtract all terms of the form $f^j(z)\nabla f^{j+n}(z+ry)$, getting

$$\begin{aligned}
\nabla f^{2n+1}_{z,r}(y) &= \sum_{j=1}^n f^j(z+ry)\nabla f^{j+n}(z+ry) - f^{j+n}(z+ry)\nabla f^j(z+ry) \\
&= \sum_{j=1}^n \big(f^j(z+ry) - f^j(z)\big)\, \nabla f^{j+n}(z+ry) - \big(f^{j+n}(z+ry) - f^{j+n}(z)\big)\, \nabla f^j(z+ry) \\
&+ \sum_{j=1}^n f^j(z)\nabla f^{j+n}(z+ry) - f^{j+n}(z)\nabla f^j(z+ry) \; .
\end{aligned}$$

Dividing by r, we can rewrite the previous equation as follows

$$\frac{1}{r}\left\{\nabla f_{z,r}^{2n+1}(y)-\sum_{j=1}^{n}\left(f^j(z)\nabla f^{j+n}(z+ry)-f^{j+n}(z)\nabla f^j(z+ry)\right)\right\}$$

$$=\sum_{j=1}^{n} f_{z,r}^{j}(y)\nabla f^{j+n}(z+ry)-f_{z,r}^{j+n}(y)\nabla f^j(z+ry)\ .$$

Since $Df(z+ry)=Df_{z,r}(y)$, this immediately leads to the conclusion. □

Next, we show that, under sufficient integrability conditions, it is possible to take somehow the differential of both sides of (1.7), achieving the following theorem.

Lemma 4.2. *Let $f\in W^{1,1}(\Omega,\mathbb{R}^{2n+1})$ and assume that a.e. we have*

$$df^{2n+1}=\sum_{j=1}^{n}\left(f^j df^{j+n}-f^{j+n}df^j\right). \tag{4.2}$$

Then $\sum_{j=1}^{n} df^j\wedge df^{j+n}=0$ holds a.e. in Ω.

Proof. Let $z\in\Omega$ be a Lebesgue point of both f and Df. We choose $\rho_h\searrow 0$ such that $\rho_1<\rho$ and set $u_h=f_{z,\rho_h}$. By Lemma 4.1, there exists $w_h\in W^{1,1}(\mathbb{B})$ such that for $\mathcal{L}^2$-almost every $y\in\mathbb{B}$ we have

$$dw_h(y)=\sum_{j=1}^{n} u_h^j(y)\,du_h^{j+n}(y)-u_h^{j+n}(y)\,du_h^j(y)\ .$$

Furthermore, since z is a Lebesgue point of both f and Df, it follows that

$$u_h\to u \ \text{ in } \ W^{1,1}(\mathbb{B}),\quad \text{where}\quad u(y)=Df(z)\cdot y,\quad y\in\mathbb{B}. \tag{4.3}$$

We may assume that the sequence ρ_h is defined in such a way that the convergence in (4.3) is fast. Lemma 3.1 implies that for almost every $r\in(0,1)$ the integral

$$\int_{\partial B(0,r)}\Big(\sum_{j=1}^{n} u_h^j\,du_h^{j+n}-u_h^{j+n}\,du_h^j\Big)$$

is well defined and equal to $\int_{\partial B(0,r)} dw_h$. Thus, in view of Lemma 3.2 we have

$$\int_{\partial B(0,r)}\Big(\sum_{j=1}^{n} u_h^j\,du_h^{j+n}-u_h^{j+n}\,du_h^j\Big)=\int_{\partial B(0,r)} dw_h=0$$

for all h and almost every $r\in(0,1)$. Taking into account both (4.3) and Lemma 3.1, for almost every $r\in(0,1)$ we have

$$0=\int_{\partial B(0,r)}\Big(\sum_{j=1}^{n} u_h^j\,du_h^{j+n}-u_h^{j+n}\,du_h^j\Big)\to\int_{\partial B(0,r)}\Big(\sum_{j=1}^{n} u^j\,du^{j+n}-u^{j+n}\,du^j\Big).$$

It is enough to pick one such a radius, so that by Stokes theorem, we obtain

$$\int_{B(0,r)} \sum_{j=1}^{n} du^j \wedge du^{j+n} = 0. \tag{4.4}$$

The equation (4.4) yields

$$\mathcal{L}^2(B(0,r)) \sum_{j=1}^{n} \det\big(\nabla f^j(z), \nabla f^{j+n}(z)\big) = 0\,.$$

Thus, we have $\sum_{j=1}^{n} \det\big(\nabla f^j(z), \nabla f^{j+n}(z)\big) = 0$. Since Lebesgue points of both f and Df have full measure in Ω, our claim follows. □

5. The m-dimensional case

In this section we treat the general case $m \geq 2$.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^m$ be open and let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$. If (1.7) holds a.e. in Ω, then*

$$\sum_{j=1}^{n} df^j \wedge df^{j+n} = 0 \tag{5.1}$$

holds a.e. in Ω.

Proof. It is enough to verify (5.1) on an arbitrary m-dimensional open cube $Q \subset\subset \Omega$. Fix $1 \leq k < l \leq m$. We set $\Gamma = \{k, l\}$ and use the notation of Section 2, with the exception that now we use the *subscript* z to denote the section

$$f_z(y) = f(z + y), \qquad y \in Q_\Gamma.$$

By Proposition 2.2, for a.e. $z \in \hat{Q}_\Gamma$ we have that $f_z \in W^{1,1}(Q_\Gamma)$ and

$$\frac{\partial f_z}{\partial x_k} = \left(\frac{\partial f}{\partial x_k}\right)_z, \qquad \frac{\partial f_z}{\partial x_l} = \left(\frac{\partial f}{\partial x_l}\right)_z \quad \text{a.e. in } Q_\Gamma. \tag{5.2}$$

In particular, we have

$$df_z^{2n+1} = \sum_{j=1}^{n} \left(f_z^j df_z^{j+n} - f_z^{j+n} df_z^j\right) \quad \text{a.e. in } Q_\Gamma.$$

Then use Lemma 4.2 on Q_Γ to infer that

$$\sum_{j=1}^{n} df_z^j \wedge df_z^{j+n} = 0 \quad \text{a.e. in } Q_\Gamma.$$

Using Fubini's theorem and (5.2) we obtain that

$$\sum_{j=1}^{n} \det \begin{pmatrix} \frac{\partial f^j}{\partial x_k}, & \frac{\partial f^j}{\partial x_l} \\ \frac{\partial f^{j+n}}{\partial x_k}, & \frac{\partial f^{j+n}}{\partial x_l} \end{pmatrix} = 0 \quad \text{a.e. in } Q.$$

By the arbitrary choice of k and l, the equality (5.1) holds a.e. in Q. □

6. Non-horizontality of higher dimensional Sobolev sets

We begin this section with the following algebraic lemma.

Lemma 6.1. *Let m, n be positive integers and let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n} \in \mathbb{R}^m$. If we have*

$$\sum_{j=1}^{n} \boldsymbol{u}_j \wedge \boldsymbol{u}_{j+n} = 0,$$

then the matrix B with rows $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n}$ has rank at most n.

Proof. We denote the inner product in $\mathbb{R}^{2n}$ by $\langle \cdot, \cdot \rangle$. Further, $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{2n})$ is the canonical basis of $\mathbb{R}^{2n}$ and I_n is the $n \times n$ identity matrix. We consider the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Choose $\boldsymbol{v} = (v_1, \dots, v_m)$, $\boldsymbol{w} = (w_1, \dots, w_m) \in \mathbb{R}^m$. We have

$$B\boldsymbol{w} = \sum_{i=1}^{n} \sum_{k=1}^{m} (b_i^k w_k \boldsymbol{e}_i + b_{i+n}^k w_k \boldsymbol{e}_{i+n}) \ , \qquad JB\boldsymbol{v} = \sum_{j=1}^{n} \sum_{l=1}^{m} (b_j^l v_l \boldsymbol{e}_{j+n} - b_{j+n}^l v_l \boldsymbol{e}_j)$$

and this implies that

$$\langle B\boldsymbol{w},\ JB\boldsymbol{v} \rangle = \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \langle b_i^k w_k \boldsymbol{e}_i + b_{i+n}^k w_k \boldsymbol{e}_{i+n},\ b_j^l v_l \boldsymbol{e}_{j+n} - b_{j+n}^l v_l \boldsymbol{e}_j \rangle.$$

The summands are nonzero only for $i = j$, in which case

$$\langle b_i^k w_k \boldsymbol{e}_i + b_{i+n}^k w_k \boldsymbol{e}_{i+n},\ b_i^l v_l \boldsymbol{e}_{i+n} - b_{i+n}^l v_l \boldsymbol{e}_i \rangle = w_k v_l \det \begin{pmatrix} b_i^l, & b_i^k \\ b_{i+n}^l, & b_{i+n}^k \end{pmatrix} \ ,$$

so that

$$\langle B\boldsymbol{w},\ JB\boldsymbol{v} \rangle = \sum_{k,l=1}^{m} w_k v_l \sum_{i=1}^{n} \det \begin{pmatrix} b_i^l, & b_i^k \\ b_{i+n}^l, & b_{i+n}^k \end{pmatrix} = \sum_{k,l=1}^{m} w_k v_l \Big(\sum_{i=1}^{n} \boldsymbol{u}_i \wedge \boldsymbol{u}_{i+n} \Big)_{l,k} = 0.$$

Then the images of B and of JB are orthogonal subspaces of $\mathbb{R}^{2n}$, having the same dimension, hence the rank of B cannot be greater than n. □

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^m$ be an open set and consider $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n+1})$ which satisfies (1.7) a.e. in Ω. Then Df has rank at most n a.e. in Ω.*

Proof. This is a consequence of Theorem 5.1 and Lemma 6.1. □

By Theorem 6.1, the proof of Theorem 1.2 follows essentially the same lines of [16]. Next, for the sake of the reader, we adapted this proof to our setting.

Proof of Theorem 1.2. By Theorem 6.1, the equation (1.7) fails to hold for f on a set $E \subset \Omega$ of positive $\mathcal{L}^m$-measure. We can assume that E is bounded, made of density points, that everywhere on E the approximate differential of f exists and equals its distributional differential and they have everywhere rank equal to m. Up to taking a smaller piece of E, we can also assume that f is Lipschitz. Then we consider a Lipschitz extension of

$f|_E$ to all of $\mathbb{R}^m$ and apply Whitney extension theorem, hence finding a subset E_0 of E with positive measure and $g \in C^1(\mathbb{R}^m, \mathbb{R}^{2n+1})$ such that $g|_{E_0} = f|_{E_0}$ and the approximate differential of f and the differential of g coincide on E_0. We choose $y_0 \in E_0$ and notice that for a fixed $r_0 > 0$ sufficiently small, we have $\mathcal{L}^m\big(B(y_0, r_0) \cap E_0\big) > 0$ and $\Sigma_0 = g\big(B(y_0, r_0)\big)$ is an m-dimensional embedded manifold of $\mathbb{R}^{2n+1}$. Here $B(y_0, r_0) \subset \mathbb{R}^m$ denotes the open Euclidean ball of center y_0 and radius r_0. By the properties of g and the classical area formula, we have

$$\Sigma_1 = f(B(y_0, r_0) \cap E_0) = g\big(B(y_0, r_0) \cap E_0\big) \subset \Sigma_0 \cap \Sigma \quad \text{and} \quad \mathcal{H}^m_{|\cdot|}(\Sigma_1) > 0.$$

Since (1.7) does not hold on E_0, for any $y \in B(y_0, r_0) \cap E_0$, we have $T_{f(y)}\Sigma_0 \not\subset H_y\mathbb{H}^n$, therefore

$$\tau_{\Sigma_0, \mathcal{V}}(f(y)) \neq 0,$$

where we have used the notation $\tau_{\Sigma_0,\mathcal{V}}(x)$ with $x \in \Sigma_0$ to indicate the *vertical tangent m-vector* to Σ_0 at x, see [14, Definition 2.14]. This m-vector vanishes exactly at those points x where $T_x\Sigma_0 \subset H_x\mathbb{H}^n$, see [14, Proposition 3.1]. From both [13] and [14], the spherical Hausdorff measure $\mathcal{S}^{m+1}_d \llcorner \Sigma_0$ is equivalent, up to geometric constants, to the measure $|\tau_{\Sigma_0,\mathcal{V}}|\, \mathcal{H}^m_{|\cdot|} \llcorner \Sigma_0$, hence in particular $\mathcal{S}^{m+1}_d(\Sigma_1) > 0$, therefore

$$\mathcal{H}^{m+1}_d(\Sigma) \geq \mathcal{H}^{m+1}_d(\Sigma_1) > 0\,,$$

so the proof is complete. □

6.1. **Formal horizontality of some BV graphs.** By the arguments in the proof of Theorem 1.2, it is not difficult to establish a kind of "generalized horizontal tangency" of BV functions whose graph satisfies the metric constraint (1.3). In the following theorem, $\mathbb{H}^1$ is again identified with $\mathbb{R}^3$ equipped with the vector fields (1.1) and d denotes the corresponding SR-distance.

Theorem 6.2. *Let $2 \leq \alpha < 3$ and let $g : (0,1)^2 \to \mathbb{R}$ be a BV function such that its graph*

$$G = \{(x_1, x_2, g(x)) :\ 0 < x_1, x_2 < 1\} \quad \textit{satisfies} \quad \mathcal{H}^\alpha_d(G) < +\infty\,.$$

Then the approximate gradient of g almost everywhere satisfies

$$\operatorname{ap}\nabla g(x) = (-x_2, x_1)\,. \tag{6.1}$$

Remark 6.2. As already mentioned in the introduction, the existence of BV functions that satisfy the assumptions of Theorem 6.2 with $\alpha = 2$ has been proved by Z. M. Balogh, R. Hoefer-Isenegger and J. T. Tyson, [4]. The existence of BV functions whose absolutely continuous part of the distributional gradient almost everywhere equals a vector field with nonvanishing curl is a special instance of a general result due to G. Alberti, [1].

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